

# Thermodynamics of the 3-State Potts Spin Chain

Rinat Kedem<sup>1</sup>

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We demonstrate the relation of the infrared momentum restrictions of conformal field theory with entropy considerations of finite-temperature thermodynamics for the 3-state Potts chain. We compute the free energy and obtain the low-temperature specific heat for both the ferromagnetic and antiferromagnetic spin chains, and find the central charges for both.

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**KEY WORDS:** 3-state Potts; Bethe ansatz; quasiparticles; central charge; specific heat; infrared anomaly; spin chain.

## 1. INTRODUCTION

The conformal field theory treatment of quantum spin chains at positive temperature  $T$  deals with systems of size  $M$  in the limit

$$M \rightarrow \infty, \quad T \rightarrow 0, \quad TM \text{ fixed} \quad (1.1)$$

This limit is discussed in terms of a variable  $q = \exp(2\pi v/MT)$ , where  $v$  is the speed of sound. The modular invariant partition function is computed in terms of  $q$ , and one of the important results<sup>(1,2)</sup> is that as  $q \rightarrow 1$ , the free energy per site is given as

$$f = e_{\text{GS}} - \frac{c\pi}{6v} T^2 + o(T^2) \quad (1.2)$$

where  $c$  is the central charge as determined from finite-size corrections to the ground-state energy

$$E_{\text{GS}} = Me_0 - \frac{\pi cv}{6M} + o(M^2) \quad (1.3)$$

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<sup>1</sup> Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794.

However, the limit (1.1) is not the limit of thermodynamics. This limit is

$$M \rightarrow \infty, \quad T \text{ fixed} \quad (1.4)$$

Here the low-temperature behavior of the specific heat is obtained by letting  $T \rightarrow 0$  after the limit (1.4) is taken.

These two limits will give the same result if there are no additional length scales in the problem. In this case the result is obtained<sup>(1,2)</sup> that the specific heat  $C$  is

$$C \sim \frac{\pi c}{3v} T \quad (1.5)$$

For the antiferromagnetic 3-state Potts chain, the low-lying order-one excitations in the limit (1.1) were used in ref. 3 to compute the partition function, and the result (1.5) was obtained. This was accomplished by adding up the order-one excitations found from the Bethe equations for the model.<sup>(4,5)</sup> The partition function obtained in this manner is the modular invariant partition function of conformal field theory.

The counting of states in ref. 3 depends on the fact that the momenta of order-one excitations not only obey a Fermi exclusion rule  $P_j \neq P_k$ , but also have additional exclusion rules: The number of states near  $e(P) = 0$  diminishes as the number of order-one excitations in the system increases. Correct counting of the states, incorporating these exclusion rules, gives a partition function which has a central charge smaller than that of fermions. In the case of the antiferromagnetic 3-state Potts model, there are three kinds of quasiparticle excitations, which, for purely fermionic exclusion rules, would give a central charge of  $3/2$ , whereas the actual central charge of the model is 1.

In this paper, we use the thermodynamic limit (1.4) to obtain the low-temperature specific heat of the 3-state Potts chain from Bethe's equations, using the methods of refs. 6–8. In Section 2, we write the Bethe equations for the finite lattice and introduce the completeness rules<sup>(9)</sup> for  $Q = 0$ . We find it convenient to study the ferromagnetic and antiferromagnetic cases using two different sets of integral equations. In Section 3, we write the free energy for the ferromagnetic case in terms of only one integral equation, and compute the linear term in the low-temperature specific heat. In Section 4, we do the same for the antiferromagnetic chain, this time in terms of two integral equations. We obtain the central charge of the conformal limit of both spin chains from the linear term in the specific heat, using Eq. (1.5). In Section 5, we consider the sector  $Q = 1$ .

In Section 6, we discuss how the counting of states in the finite-size

system<sup>(3)</sup> is incorporated into the discussion in terms of densities in the thermodynamic limit. We find that in the thermodynamic limit, the counting of states is contained in the entropy, written in terms of densities. The densities are related through the thermodynamic limit of the Bethe equations (3.1), (4.1), and this relationship corresponds to the way the number of available states depends on the number of excitations in the finite-size system. These equations become particularly important when evaluating the low-temperature specific heat, where we need to consider the equations in precisely the limit which corresponds to  $P \rightarrow 0$  in order to extract the linear term in the specific heat.

It is important to note that the 3-state Potts model is the  $D_4$  model in the classification of Pasquier,<sup>(10)</sup> which, by orbifold construction,<sup>(11,12)</sup> has certain sectors of eigenvalues which overlap with the  $A_5$  RSOS model. From this construction, it is to be expected that the thermodynamic quantities of the two models are equal as long as the sectors which dominate the thermodynamics are common to the sectors which overlap. Indeed, the ground states of both the ferromagnetic and antiferromagnetic ends of the  $D_4$  model are the same as the critical  $A_5$  model at the boundaries of the III/IV and I/II regimes, respectively. Thus the central charges computed for  $A_5$  (refs. 13–15) coincide with the central charges of the  $D_4$  model.<sup>(16,17)</sup> We further note that the classical two-dimensional antiferromagnetic 3-state Potts model, which is critical at  $T=0$ , is equivalent<sup>(18–20)</sup> to the 3-coloring problem and also has central charge  $c=1$ .<sup>(24)</sup>

The thermodynamics of the  $A_n$  series was studied in ref. 15, where the integral equations for the free energy, the central charges, and order-one excitations above the ground state were found. The thermodynamics of other affine Lie algebras was also studied in ref. 21. In this paper, however, we obtain different sets of integral equations, which display a more direct relationship to the modular invariant partition function discussed in ref. 3 for the antiferromagnetic chain and in ref. 23 for the ferromagnetic chain. The fact that there are different sets of integral equations for the model is related to the fact that the modular invariant partition function can be expressed as sums over different sets of quasiparticle excitations.<sup>(22)</sup>

## 2. FORMULATION

The three-state Potts Hamiltonian is

$$H = \pm \frac{2}{\sqrt{3}} \sum_{j=1}^M \{X_j + X_j^\dagger + Z_j Z_{j+1}^\dagger + Z_j^\dagger Z_{j+1}\} \quad (2.1)$$

with periodic boundary conditions, where  $M$  is the number of sites in the chain, and the matrices  $X_j$  and  $Z_j$  are

$$\begin{aligned} X_j &= I \otimes I \otimes \cdots \otimes X_{j\text{th}} \otimes \cdots \otimes I \\ Z_j &= I \otimes I \otimes \cdots \otimes Z_{j\text{th}} \otimes \cdots \otimes I \end{aligned} \tag{2.2}$$

$I$  is the  $3 \times 3$  identity matrix and  $X$  and  $Z$  are  $3 \times 3$  matrices with entries

$$X_{ij} = \delta_{i,j+1 \pmod{3}}, \quad Z_{ij} = \delta_{i,j} \omega^{i-1}, \quad \omega = e^{2\pi i/3} \tag{2.3}$$

The Hamiltonian with the (+) – sign is referred to as the (anti)-ferromagnetic Potts chain. It commutes with the spin rotation operator, whose eigenvalues are  $e^{2\pi i Q/3}$ ,  $Q = 0, \pm 1$ . The eigenvalues of the Hamiltonian (2.1) are derived from functional equations<sup>(13,15,25–27)</sup> from which we find<sup>(4)</sup>

$$E = \sum_{j=1}^L \cot \left( i\lambda_j + \frac{\pi}{12} \right) - \frac{2M}{\sqrt{3}}, \quad L = 2(M - |Q|), \quad Q = 0, 1, -1 \tag{2.4}$$

where the set  $\{\lambda_j\}$  satisfy the Bethe equations:

$$\begin{aligned} & \left[ \frac{\sinh(i\pi/12 - \lambda_j)}{\sinh(i\pi/12 + \lambda_j)} \right]^{2M} \\ &= (-1)^{M+1} \prod_{k=1}^L \frac{\sinh(i\pi/3 - (\lambda_j - \lambda_k))}{\sinh(i\pi/3 + (\lambda_j - \lambda_k))}, \quad j = 1, \dots, L \end{aligned} \tag{2.5}$$

Not all solutions of Eqs. (2.5) correspond to eigenvalues of the Hamiltonian (2.1). The equations do not impose sufficient restrictions on the set  $\{\lambda_j\}$ . There are additional conditions, which ensure that the energy is real, postulated from finite-size studies in ref. 9, where the spectrum of  $H$  was classified. We introduce these conditions by writing (2.5) in logarithmic form. The solutions of (2.5) which correspond to eigenstates of the Hamiltonian (2.1) fall into five classes, where below  $\lambda_j$  is a real number:

$$\begin{aligned} \lambda_j^+ &= \lambda_j, & \lambda_j^- &= \lambda_j + \frac{i\pi}{2}, & \lambda_j^{2s} &= \lambda_j \pm \frac{i\pi}{6} \\ \lambda_j^{-2s} &= \lambda_j \pm \frac{i\pi}{3}, & \lambda_j^{ns} &= \lambda_j \pm \frac{i\pi}{4} \end{aligned} \tag{2.6}$$

The last three always occur in complex conjugate pairs. Since we are interested in the infinite lattice limit, the imaginary parts are assumed to be exact. We denote the number of each type of root  $\alpha \in \{+, -, 2s, -2s, ns\}$  by  $m_\alpha$ .

We rewrite the Bethe equations (2.5) to display explicitly the different types of roots (2.6). Let

$$h(\lambda) = \frac{\sinh(i\pi/3 - \lambda)}{\sinh(i\pi/3 + \lambda)} \tag{2.7}$$

Then (2.5) become, for  $\alpha = +$  or  $-$ ,

$$\begin{aligned} & (-1)^{M+1} \left[ \frac{\sinh(i\pi/12 - \lambda_j^\alpha)}{\sinh(i\pi/12 + \lambda_j^\alpha)} \right]^{2M} \\ &= \prod_{k=1}^{m_+} h(\lambda_j^\alpha - \lambda_k^+) \prod_{k=1}^{m_-} h(\lambda_j^\alpha - \lambda_k^-) \prod_{k=1}^{m_{2s}} h(\lambda_j^\alpha - \lambda_k^{2s}) h(\lambda_j^\alpha - \lambda_k^{2s*}) \\ &\quad \times \prod_{k=1}^{m_{-2s}} h(\lambda_j^\alpha - \lambda_k^{-2s}) h(\lambda_j^\alpha - \lambda_k^{-2s*}) \prod_{k=1}^{m_{ns}} h(\lambda_j^\alpha - \lambda_k^{ns}) h(\lambda_j^\alpha - \lambda_k^{ns*}) \end{aligned} \tag{2.8}$$

whereas for  $\alpha = 2s, -2s,$  or  $ns,$  the equations for each complex conjugate pair are multiplied together:

$$\begin{aligned} & \left[ \frac{\sinh(i\pi/12 - \lambda_j^\alpha) \sinh(i\pi/12 - \lambda_j^{\alpha*})}{\sinh(i\pi/12 + \lambda_j^\alpha) \sinh(i\pi/12 + \lambda_j^{\alpha*})} \right]^{2M} \\ &= \prod_{k=1}^{m_+} h(\lambda_j^\alpha - \lambda_k^+) h(\lambda_j^{\alpha*} - \lambda_k^+) \prod_{k=1}^{m_-} h(\lambda_j^\alpha - \lambda_k^-) h(\lambda_j^{\alpha*} - \lambda_k^-) \\ &\quad \times \prod_{k=1}^{m_{2s}} h(\lambda_j^\alpha - \lambda_k^{2s}) h(\lambda_j^{\alpha*} - \lambda_k^{2s}) h(\lambda_j^\alpha - \lambda_k^{2s*}) h(\lambda_j^{\alpha*} - \lambda_k^{2s*}) \\ &\quad \times \prod_{k=1}^{m_{-2s}} h(\lambda_j^\alpha - \lambda_k^{-2s}) h(\lambda_j^{\alpha*} - \lambda_k^{-2s}) h(\lambda_j^\alpha - \lambda_k^{-2s*}) h(\lambda_j^{\alpha*} - \lambda_k^{-2s*}) \\ &\quad \times \prod_{k=1}^{m_{ns}} h(\lambda_j^\alpha - \lambda_k^{ns}) h(\lambda_j^{\alpha*} - \lambda_k^{ns}) h(\lambda_j^\alpha - \lambda_k^{ns*}) h(\lambda_j^{\alpha*} - \lambda_k^{ns*}) \end{aligned} \tag{2.9}$$

We follow ref. 9 in taking the logarithm of Eqs. (2.8), (2.9). To do this, we define the functions  $t_\alpha$  and  $\Theta_{\alpha,\beta}$  in the following way:

$$t_\alpha(\lambda_j^\alpha) = \begin{cases} -2i \ln \left[ \pm \frac{\sinh(i\pi/12 - \lambda_j^\pm)}{\sinh(i\pi/12 + \lambda_j^\pm)} \right] & \text{for } \alpha = \pm \\ -2if_\alpha \ln \left[ \frac{\sinh(i\pi/12 - \lambda_j^\alpha) \sinh(i\pi/12 - \lambda_j^{\alpha*})}{\sinh(i\pi/12 + \lambda_j^\alpha) \sinh(i\pi/12 + \lambda_j^{\alpha*})} \right] & \text{for } \alpha = \pm 2s, ns \end{cases} \tag{2.10}$$

where  $f_{\pm 2s} = -1$ ,  $f_{ns} = -1/2$ , and  $f_{\pm} = 1$ . For  $\alpha, \beta = \pm$

$$\Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) = -i \ln[\alpha\beta h(\lambda_j^\alpha - \lambda_k^\beta)] \tag{2.11}$$

For  $\alpha = \pm, \beta = \pm 2s, ns$  or  $\alpha = \pm 2s, ns, \beta = \pm,$

$$\Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) = -if_\alpha \ln[\varepsilon_{\alpha\beta} h(\lambda_j^\alpha - \lambda_k^\beta) h(\lambda_j^\alpha - \lambda_k^{\beta*})] \tag{2.12}$$

where  $\varepsilon_{+, -2s} = \varepsilon_{-, 2s} = -1, = 1$  otherwise. For  $\alpha, \beta = \pm 2s, ns$

$$\Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) = -if_\alpha \ln[\varepsilon_{\alpha\beta} h(\lambda_j^\alpha - \lambda_k^\beta) h(\lambda_j^\alpha - \lambda_k^{\beta*}) h(\lambda_j^{\alpha*} - \lambda_k^\beta) h(\lambda_j^{\alpha*} - \lambda_k^{\beta*})] \tag{2.13}$$

where  $\varepsilon_{2s, 2s} = \varepsilon_{-2s, -2s} = -1, 1$  otherwise. (Note that in ref. 9 the functions  $t_\alpha$  and  $\Theta_{\alpha\beta}$  for  $\alpha = \pm 2s, ns$  were defined without the factor  $f_\alpha$ . This will change the completeness rules somewhat from those presented in ref. 9, but is necessary in order to have positive densities.) Here, all logarithms are taken so that  $|\text{Im} \ln z| \leq \pi$ , and the functions  $t_\alpha$  and  $\Theta_{\alpha\beta}$  are defined so that

$$\begin{aligned} t_\alpha(\lambda_j^\alpha) &= 0 & \text{if } \text{Re}(\lambda_j^\alpha) &= 0 \\ \Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) &= 0 & \text{if } \text{Re}(\lambda_j^\alpha) &= \text{Re}(\lambda_k^\beta) \end{aligned} \tag{2.14}$$

The logarithmic Bethe equations are written in terms of these functions:

$$Z(\lambda_j^\alpha) \equiv \frac{I_j^\alpha}{M} = \frac{1}{2\pi} t_\alpha(\lambda_j^\alpha) - \frac{1}{2\pi M} \sum_{\beta = \pm, \pm 2s, ns} \sum_{k=1}^{m_\beta} \Theta_{\alpha\beta}(\lambda_j^\alpha - \lambda_k^\beta) \tag{2.15}$$

where  $I_j^\alpha$  are (half-) integers. We now present the completeness rules for the  $I_j^\alpha$ . It is only necessary at this point to discuss the completeness rules for  $Q = 0$ . It will be shown in Section 5 that the  $Q = \pm 1$  sectors are identical to this sector in the thermodynamic limit. The completeness rules of ref. 9 for  $Q = 0$  in the notation introduced here become:

1.  $I_k^+$  and  $I_k^{2s}$  are distinct (half-) integers, are chosen from the same set of  $m_+ + m_{2s}$  (half-) integers, and  $I_k^+ = I_k^{2s h}$ , where  $h$  represents a ‘‘hole’’ or missing (half-) integer. Therefore the set  $\{I_j^+\} + \{I_j^{2s}\}$  fills the interval  $-1/2(m_+ + m_{2s-1})$  to  $1/2(m_+ + m_{2s-1})$ .
2.  $I_k^-$  and  $I_k^{-2s}$  are distinct (half-) integers, are chosen from the same set of  $m_- + m_{-2s}$  (half-) integers, and  $I_k^- = I_k^{-2s h}$ . Again, the set  $\{I_j^-\} + \{I_j^{-2s}\}$  fills the interval.
3.  $I_k^{ns}$  are distinct (half-) integers chosen from a set of  $2m_- + 2m_{-2s} + m_{ns}$  (half-) integers.
4. The spacing between ‘‘available’’ integers, the set of integers  $\{I_k^\alpha\} + \{I_k^{\alpha h}\}$ , is 1.

We see that + integers correspond to missing  $2s$  integers, and the same for - and  $-2s$ . In addition to these rules, there is a sum rule for  $m_x$ :

$$\begin{aligned} m_+ &= 2n_{ns} + 3m_- + 4m_{-2s} \\ m_{2s} + 2m_{ns} + 3m_{-2s} + 2m_- &= M \end{aligned} \tag{2.16}$$

This sum rule is responsible for restricting the maximum integers  $I_{\max}^\alpha$  as a function of the number of excitations in the system.

We make the assumption that at large  $M$  the rules 1 and 2 imply the equalities

$$\lambda_j^+ = \lambda_j^{2s h}, \quad \lambda_j^- = \lambda_j^{-2s h} \tag{2.17}$$

This appears to be true from numerical results and has been proven for order-one excitations,<sup>(5)</sup> and has been shown to be consistent for all excitation densities in the thermodynamic limit.

We now take the thermodynamic limit  $M \rightarrow \infty$  of the Bethe equations (2.15), with  $\lambda$  fixed. When we do this we lose the information contained in the rules 1-3 about the maximum integers. We rewrite the functions  $\Theta_{\alpha, \beta}$  and  $t_\alpha$  in terms of the real part of  $\lambda_j^\alpha$ , using (2.6), and take the derivative of  $Z(\lambda)$  with respect to  $\lambda$  in the thermodynamic limit. We obtain the following set of equations:

$$\begin{aligned} \rho_i^\pm(\lambda) &= \frac{1}{\pi} K_{\pi/12}^\pm(\lambda) - \frac{1}{2\pi} [K_{\pi/3}^\pm * (\rho_p^+ - \rho_p^{-2s}) + K_{\pi/3}^\mp * (\rho_p^- - \rho_p^{2s}) \\ &\quad + \{K_{\pi/12}^+ + K_{\pi/12}^-\} * \rho_p^{ns}] \\ \rho_i^{\pm 2s}(\lambda) &= \frac{1}{\pi} [K_{\pi/12}^\pm(\lambda) - K_{\pi/4}^\pm(\lambda)] \\ &\quad - \frac{1}{2\pi} [K_{\pi/3}^\mp * (\rho_p^+ - \rho_p^{-2s}) + K_{\pi/3}^\pm * (\rho_p^- - \rho_p^{2s}) \\ &\quad + \{K_{\pi/12}^+ + K_{\pi/12}^-\} * \rho_p^{ns}] \\ \rho_i^{ns}(\lambda) &= -\frac{1}{2\pi} [K_{\pi/3}^+(\lambda) + K_{\pi/3}^-(\lambda)] \\ &\quad + \frac{1}{2\pi} \left[ \frac{1}{2} (K_{\pi/12}^+ + K_{\pi/12}^-) * (\rho_p^+ - \rho_p^{-2s} + \rho_p^- - \rho_p^{2s}) \right. \\ &\quad \left. + (K_{\pi/3}^+ + K_{\pi/3}^-) * \rho_p^{ns} \right] \end{aligned} \tag{2.18}$$

where

$$\rho_i^\alpha = \lim_{M \rightarrow \infty} \frac{1}{M(\lambda_{I_j+1} - \lambda_{I_j})}, \quad \rho_p^\alpha = \lim_{M \rightarrow \infty} \frac{1}{M(\lambda_{I_j+1} - \lambda_{I_j})} \quad (2.19)$$

the convolution  $*$  is defined as

$$f * g = \int_{-\infty}^{\infty} d\mu f(\lambda - \mu) g(\mu) \quad (2.20)$$

and the kernels  $K_\alpha^\pm(\lambda)$  are

$$K_\alpha^\pm(\lambda) = \frac{\pm 2 \sin 2\alpha}{\cosh 2\lambda \mp \cos 2\alpha} \quad (2.21)$$

In writing Eqs. (2.18), we did not make use of the relationship between holes and integers (2.17), which imposes a relationship between the densities in Eqs. (2.18). The assumption (2.17) implies that when particle integers of, say,  $+$  are equal to the “hole” integers of  $2s$ , their corresponding rapidities are equal. Therefore, in light of the definitions (2.19), the total densities of  $+$  and  $2s$  are equal (and those of  $-$  and  $-2s$  as well), and the particle densities are related in a simple way:

$$\rho_i^+(\lambda) = \rho_i^{2s}(\lambda), \quad \rho_i^-(\lambda) = \rho_i^{-2s}(\lambda), \quad \rho_p^{\pm 2s}(\lambda) = \rho_i^\pm(\lambda) - \rho_p^\pm(\lambda) \quad (2.22)$$

This allows us to rewrite the density equations (2.18) in terms of three independent particle densities. It is convenient for further computation do this separately for the ferromagnetic and antiferromagnetic spin chains.

### 3. FERROMAGNETIC CHAIN

For the ferromagnetic Hamiltonian, we know from refs. 9 and 5 that the independent order-one excitations can be chosen to be  $+$ ,  $-$ , and  $ns$ . We therefore choose to rewrite (2.18), using (2.22), as

$$\begin{aligned} \rho_i^+(\lambda) &= \frac{6}{\pi \cosh 6\lambda} + K_1 * (\rho_p^+ + \rho_p^-) - K_2 * \rho_p^{ns} \\ \rho_i^-(\lambda) &= K_1 * (\rho_p^+ + \rho_p^-) - K_2 * \rho_p^{ns} \\ \rho_i^{ns}(\lambda) &= K_2 * (\rho_p^+ + \rho_p^-) \end{aligned} \quad (3.1)$$

where the kernels are

$$K_1(\lambda) = \frac{18}{\pi^2} \frac{\lambda}{\sinh 6\lambda}, \quad K_2(\lambda) = \frac{3}{\pi \cosh 6\lambda} \quad (3.2)$$



The particle densities in Eqs. (3.1) are now independent of each other; there are no additional constraints.

In the thermodynamic limit, the sum rule (2.16) becomes a relationship between total particle densities  $D_\alpha$ ,

$$D_\alpha = \lim_{M \rightarrow \infty} \frac{m_\alpha}{M} = \int d\lambda \rho_p^\alpha \tag{3.3}$$

However, we find that we do not need to impose the sum rule as an additional restriction on the densities in (3.1), as it is contained in those equations already. To see this, we take the Fourier transform of the first two equations in (3.1) and evaluate at  $k=0$ . This gives exactly the relationship (2.16) divided by  $M$ .

In ref. 3 the sum rules (2.16) were found to give rise to the infrared momentum restrictions, that is, to the diminishing of the number of states near  $P=0$  for the antiferromagnetic case, and thus to exclusion rules beyond those of fermions. Here, although we lose information about how the maximum integers change as a function of  $m_\alpha$  when we take the thermodynamic limit, we still retain a restriction between the densities which contains some of this information. This restriction will allow us to retain the concept of correct counting of states in the thermodynamic limit.

The free energy is

$$F = E - TS \tag{3.4}$$

evaluated at the stationary point with respect to independent particle densities, where  $S$  is the entropy of a state with fixed densities  $\rho_p^\alpha$  and  $E$  is the total energy of the state. For large  $M$  the entropy is

$$S = M \sum_{\alpha = +, -, ns} \int_{-\infty}^{\infty} d\lambda (\rho_i^\alpha \ln \rho_i^\alpha - \rho_p^\alpha \ln \rho_p^\alpha - \rho_h^\alpha \ln \rho_h^\alpha) \tag{3.5}$$

where  $\rho_h = \rho_i - \rho_p$ . The energy  $E$  is the thermodynamic limit of Eq. (2.4):

$$E = M \sum_{\alpha = +, -, 2s, -2s, ns} \int d\lambda \rho_p^\alpha(\lambda) e^\alpha(\lambda) - \frac{2M}{\sqrt{3}} \tag{3.6}$$

where  $e^\alpha(\lambda)$  is the energy associated with a root of type  $\beta$ :

$$e^\pm(\lambda) = \frac{\pm 1 - 2i \sinh 2\lambda}{2 \cosh 2\lambda \mp \sqrt{3}}, \quad e^{ns}(\lambda) = \frac{-2\sqrt{3} - 4i \sinh 4\lambda}{1 + 2 \cosh 4\lambda} \tag{3.7}$$

$$e^{\pm 2s}(\lambda) = \frac{\mp 1 - 2i \sinh 2\lambda}{2 \cosh 2\lambda \mp \sqrt{3}} + \frac{\pm 1 - i \sinh 2\lambda}{\cosh 2\lambda}$$

The energy in (3.6) is not manifestly real. However, using (3.1), we find that the energy can be reexpressed in terms of only the independent particle densities, and depends only on  $\rho_p^+$ :

$$E = -\frac{2M}{\sqrt{3}} + \int_{-\infty}^{\infty} d\lambda \frac{6}{\cosh 6\lambda} \rho_p^+(\lambda) \tag{3.8}$$

This expression for the energy is manifestly real. Minimizing (3.4) with respect to the three particle densities  $\rho_p^+$ ,  $\rho_p^-$ ,  $\rho_p^{ns}$ , we obtain the free energy per site,

$$f = e_0 - T \int_{-\infty}^{\infty} d\lambda \frac{6}{\pi \cosh 6\lambda} \ln(1 + e^{-\varepsilon^+(\lambda)/T}) \tag{3.9}$$

where

$$e_0 = -\frac{2}{\sqrt{3}} + \int_{-\infty}^{\infty} d\lambda \frac{6e^{2s}(\lambda)}{\pi \cosh 6\lambda} = -\frac{4}{\pi} - \frac{8}{3\sqrt{3}} = -2.81284... \tag{3.10}$$

and the densities  $e^\beta$  are defined as  $\rho_h^\beta/\rho_p^\beta = \exp(e^\beta/T)$ , and satisfy the nonlinear integral equations

$$\begin{aligned} \varepsilon^+(\lambda) &= \frac{6}{\cosh 6\lambda} - T[K_1 * \ln[(1 + e^{-\varepsilon^+/T})(1 + e^{-\varepsilon^-/T})] \\ &\quad + K_2 * \ln(1 + e^{-\varepsilon^{ns}/T})] \\ \varepsilon^-(\lambda) &= \varepsilon^+(\lambda) - \frac{6}{\cosh 6\lambda} \\ \varepsilon^{ns}(\lambda) &= TK_2 * [\ln(1 + e^{-\varepsilon^+(\mu)/T})(1 + e^{-\varepsilon^-(\mu)/T})] \end{aligned} \tag{3.11}$$

The functions  $\varepsilon^\alpha$  are also referred to as dressed energies. Note that (3.11) represents only one integral equation for  $\varepsilon^+$ , since  $\varepsilon^-$  is simply related to  $\varepsilon^+$ , and the equation for  $\varepsilon^{ns}$  is not an integral equation, as  $\varepsilon^{ns}$  does not appear on the right-hand side.

At fixed  $\lambda$  and  $T=0$ , we get from (3.11)

$$\varepsilon_0^+(\lambda) = \frac{6}{\cosh 6\lambda}, \quad \varepsilon_0^- = \varepsilon_0^{ns} = 0 \tag{3.12}$$

which are the order-one excitation energies found in ref. 5 for the ferromagnetic chain. The free energy per site in this limit is  $f = e_0$ , which is the ground-state energy found in refs. 28, 29, and 4.

The linear  $T$  term in the specific heat  $C$  is obtained from the low-temperature expansion of the free energy, or the entropy (3.5):

$$C = -T \frac{\partial^2 f}{\partial T^2} = T \frac{\partial S}{\partial T} \tag{3.13}$$

We find the linear term in the specific heat by computing the  $O(T)$  term in the low-temperature entropy.<sup>(30–32,15)</sup> As  $T \rightarrow 0$ ,  $\varepsilon^+(\lambda)/T$  scales as  $1/(T \cosh 6\lambda)$ , which makes no contribution to the integral (3.9) except when  $\lambda \sim O(\frac{1}{6} \ln T)$ . We rescale Eqs. (3.11) by making the change of variables  $\lambda \rightarrow \lambda - \frac{1}{6} \ln T$  and consider Eqs. (3.11) at large  $\lambda$  and small  $T$ . We define, at this range of variables,  $\phi^\beta(\lambda) \equiv \varepsilon^\beta(\lambda - \frac{1}{6} \ln T)/T$ . The integral equations become

$$\begin{aligned} \phi^+(\lambda) &\simeq 12e^{-6\lambda} - K_1 * \ln[(1 + e^{-\phi^+})(1 + e^{-\phi^-})] - K_2 * \ln(1 + e^{-\phi^{ns}}) \\ \phi^-(\lambda) &\simeq -K_1 * \ln[(1 + e^{-\phi^+})(1 + e^{-\phi^-})] - K_2 * \ln(1 + e^{-\phi^{ns}}) \\ \phi^{ns}(\lambda) &\simeq K_2 * \ln[(1 + e^{-\phi^+})(1 + e^{-\phi^-})] \end{aligned} \tag{3.14}$$

Differentiating (3.14) with respect to  $\lambda$ , we obtain

$$\begin{aligned} \frac{d\phi^+}{d\lambda} &\simeq -2 \times 36e^{-6\lambda} + K_1 * \left[ \frac{\phi'^+}{1 + e^{\phi^+}} + \frac{\phi'^-}{1 + e^{\phi^-}} \right] + K_2 * \frac{\phi'^{ns}}{1 + e^{\phi^{ns}}} \\ \frac{d\phi^-}{d\lambda} &\simeq K_1 * \left[ \frac{\phi'^+}{1 + e^{\phi^+}} + \frac{\phi'^-}{1 + e^{\phi^-}} \right] + K_2 * \frac{\phi'^{ns}}{1 + e^{\phi^{ns}}} \\ \frac{d\phi^{ns}}{d\lambda} &\simeq -K_2 * \left[ \frac{\phi'^+}{1 + e^{\phi^+}} + \frac{\phi'^-}{1 + e^{\phi^-}} \right] \end{aligned} \tag{3.15}$$

where  $\phi'^\beta = d\phi^\beta/d\lambda$ .

We rescale Eqs. (3.1) in the same way. Let  $\tilde{\rho}_i^\beta(\lambda) = \rho_i^\beta(\lambda - \frac{1}{6} \ln T)$ , and recall that  $\rho_p^\beta = \rho_i^\beta/(1 + e^{\beta/T})$ . Then

$$\begin{aligned} \tilde{\rho}_i^+(\lambda) &= \frac{12T}{\pi} e^{-6\lambda} + K_1 * \left( \frac{\tilde{\rho}_i^+}{1 + e^{\phi^+}} + \frac{\tilde{\rho}_i^-}{1 + e^{\phi^-}} \right) - K_2 * \frac{\tilde{\rho}_i^{ns}}{1 + e^{\phi^{ns}}} \\ \tilde{\rho}_i^-(\lambda) &= K_1 * \left( \frac{\tilde{\rho}_i^+}{1 + e^{\phi^+}} + \frac{\tilde{\rho}_i^-}{1 + e^{\phi^-}} \right) - K_2 * \frac{\tilde{\rho}_i^{ns}}{1 + e^{\phi^{ns}}} \\ \tilde{\rho}_i^{ns}(\lambda) &= K_2 * \left( \frac{\tilde{\rho}_i^+}{1 + e^{\phi^+}} + \frac{\tilde{\rho}_i^-}{1 + e^{\phi^-}} \right) \end{aligned} \tag{3.16}$$

Comparing Eq. (3.16) to (3.15), we see that

$$\tilde{\rho}_p^\pm = -\frac{T}{6\pi} \frac{d\phi^\pm}{d\lambda} \frac{1}{1+e^{\phi^\pm}}, \quad \tilde{\rho}_p^{ns} = \frac{T}{6\pi} \frac{d\phi^{ns}}{d\lambda} \frac{1}{1+e^{\phi^{ns}}} \tag{3.17}$$

The entropy can be evaluated in this limit. The  $\lambda \rightarrow \infty$  and  $\lambda \rightarrow -\infty$  limits make the same contribution to  $S$  (and  $f$ ). Therefore we write

$$\begin{aligned} S &\simeq 2 \sum_{\beta = +, -, ns} \int_{-\infty}^{\infty} d\lambda \{ \tilde{\rho}_p^\beta \ln(1+e^{\phi^\beta}) + \tilde{\rho}_h^\beta \ln(1+e^{-\phi^\beta}) \} \\ &= 2 \times \frac{T}{6\pi} \left[ - \int_{\phi^+(-\infty)}^{\phi^+(\infty)} d\phi g(\phi) - \int_{\phi^-(-\infty)}^{\phi^-(\infty)} d\phi g(\phi) + \int_{\phi^{ns}(-\infty)}^{\phi^{ns}(\infty)} d\phi g(\phi) \right] \end{aligned} \tag{3.18}$$

where

$$g(\phi) = \frac{\ln(1+e^\phi)}{1+e^\phi} + \frac{\ln(1+e^{-\phi})}{1+e^{-\phi}} \tag{3.19}$$

The limits  $\phi(\pm\infty)$  are found from Eqs. (3.14). In these limits, the integrals can be performed by taking the log out from under the integral sign, and integrating only the kernel. Let  $\tilde{\phi}$  denote the asymptotic value under consideration,  $\phi(\lambda = \pm\infty)$ . Then, for  $\lambda = \infty$ , we obtain a system of equations

$$\begin{aligned} \tilde{\phi}^+ &= \tilde{\phi}^- = -\frac{1}{4} \ln[(1+e^{-\tilde{\phi}^+})(1+e^{-\tilde{\phi}^-})] - \frac{1}{2} \ln(1+e^{-\tilde{\phi}^{ns}}) \\ \tilde{\phi}^{ns} &= \frac{1}{2} \ln[(1+e^{-\tilde{\phi}^+})(1+e^{-\tilde{\phi}^-})] \end{aligned} \tag{3.20}$$

Therefore the upper limits of Eq. (3.18) are

$$\phi^+(\infty) = \phi^-(\infty) = -\ln 2, \quad \phi^{ns}(\infty) = \ln 3 \tag{3.21}$$

At  $\lambda = -\infty$ ,  $\tilde{\phi}^+ = \infty$ , and

$$\begin{aligned} \tilde{\phi} &= -\frac{1}{4} \ln(1+e^{-\tilde{\phi}^-}) - \frac{1}{2} \ln(1+e^{-\tilde{\phi}^{ns}}) \\ \tilde{\phi}^{ns} &= \frac{1}{2} \ln(1+e^{-\tilde{\phi}^-}) \end{aligned} \tag{3.22}$$

and thus the lower limits in (3.18) are

$$\begin{aligned} \phi^+(-\infty) &= \infty, \quad \phi^-(-\infty) = -\ln \left[ \frac{1+\sqrt{5}}{2} \right] \\ \phi^{ns}(-\infty) &= \ln \left[ \frac{1+\sqrt{5}}{2} \right] \end{aligned} \tag{3.23}$$

We show in the Appendix how to express the integral (3.18) in terms of dilogarithms. Using the identities described in the Appendix, we find that in the low-temperature limit,

$$S \simeq \frac{4\pi T}{45} \tag{3.24}$$

and from (1.5) we see that, with  $v = 3$ ,<sup>(5)</sup> the central charge  $c = 4/5$ . This is the central charge of the well-known conformal limit of the 3-state Potts chain<sup>(14,33,34)</sup> computed in the limit (1.1). This verifies that the limits (1.1) and (1.4) smoothly connect together, and there are no additional length scales in the problem.

In the calculation above, we find that, although in the zero-temperature limit the energies  $\varepsilon^-$ ,  $\varepsilon^{ns}$  vanish, they contribute to the low-temperature specific heat, i.e., the functions  $\phi^-(\lambda)$ ,  $\phi^{ns}(\lambda)$  do not vanish. This is a manifestation of the feature seen in ref. 5, that although the energy can be expressed, as in (3.8), in terms of only  $\rho_p^+$ , the number of states  $e^S$  with energy  $E$  depends on  $\rho_i^\alpha$ , and therefore on  $\rho_p^-$  and  $\rho_p^{ns}$ , as it depends on  $m_-$ ,  $m_{ns}$  in ref. 5. In our case, the densities  $\rho_p^-$ ,  $\rho^{ns}$  enter the expression for the free energy via the entropy  $S$ , which counts the states. Computation of the specific heat at low temperature depends sensitively on correct counting of states, as was seen in ref. 3 for the antiferromagnetic case. Note that from the counting rules of ref. 9 and from Eqs. (3.1) the number of states near  $P=0$  increases as the number of excitations increases. This causes the central charge to be larger than 1/2, the value we would expect if the excitations  $\rho_p^+$  were fermions.

#### 4. ANTIFERROMAGNETIC CHAIN

To find the low-temperature behavior of the antiferromagnetic chain, we rewrite (2.18) in term of the  $\pm 2s$ ,  $ns$  densities, which we know to be the order-one excitations for this Hamiltonian.<sup>(9,5,3)</sup> Equations (2.18) become

$$\begin{aligned} \rho_i^{\pm 2s}(\lambda) &= \frac{3}{\pi} \frac{1}{\sqrt{2} \cosh 3\lambda \mp 1} - \bar{K}_1 * (\rho_p^{2s} + \rho_p^{-2s}) - \bar{K}_2 * \rho_p^{ns} \\ \rho_i^{ns}(\lambda) &= \frac{3}{\pi \cosh 3\lambda} - \bar{K}_2 * (\rho_p^{2s} + \rho_p^{-2s}) - 2\bar{K}_1 * \rho_p^{ns} \end{aligned} \tag{4.1}$$

where the kernels are

$$\bar{K}_1(\lambda) = \frac{3}{2\pi \cosh 3\lambda}, \quad \bar{K}_2(\lambda) = \frac{6 \cosh 3\lambda}{\sqrt{2} \pi \cosh 6\lambda} \tag{4.2}$$

Note that, from these equations, in the antiferromagnetic case the density of available states always diminishes with increasing particle densities. This was seen in refs. 9 and 3, where the number of available states decreases as a function of  $m_{2s}$ ,  $m_{-2s}$ ,  $m_{ns}$ .

The entropy in terms of these densities looks the same as (3.5), but now we sum over the three independent densities  $\alpha = \pm 2s, ns$ . We also express  $E$  in terms of the  $\pm 2s, ns$  densities. Minimizing the quantity (3.4) with respect to the three particle densities  $\rho_p^{2s}$ ,  $\rho^{-2s}$ ,  $\rho^{ns}$  now gives the free energy in terms of these densities:

$$f = \tilde{e}_0 + T \int_{-\infty}^{\infty} d\lambda \left\{ \frac{3 \ln(1 + e^{-\varepsilon^{2s}(\lambda)/T})}{\pi(\sqrt{2} \cosh 3\lambda - 1)} + \frac{3 \ln(1 + e^{-\varepsilon^{-2s}(\lambda)/T})}{\pi(\sqrt{2} \cosh 3\lambda + 1)} + \frac{3 \ln(1 + e^{-\varepsilon^{ns}(\lambda)/T})}{\pi \cosh 3\lambda} \right\} \quad (4.3)$$

where

$$\begin{aligned} \tilde{e}_0 &= e_0 + \frac{18}{\pi^2} \int_{-\infty}^{\infty} d\lambda \frac{1}{\cosh 6\lambda(\sqrt{2} \cosh 3\lambda - 1)} \\ &= 3 - \frac{8}{3\sqrt{3}} + \frac{2}{\pi} = 2.097... \end{aligned} \quad (4.4)$$

and the  $\varepsilon^\alpha$  satisfy the integral equations

$$\begin{aligned} \varepsilon^{\pm 2s}(\lambda) &= \frac{3}{\sqrt{2} \cosh 3\lambda \mp 1} \\ &\quad + T \{ \bar{K}_1 * \ln[(1 + e^{-\varepsilon^{2s}/T})(1 + e^{-\varepsilon^{-2s}/T})] + \bar{K}_2 * \ln(1 + e^{-\varepsilon^{ns}/T}) \} \\ \varepsilon^{ns}(\lambda) &= \frac{3}{\cosh 3\lambda} \\ &\quad + T \{ \bar{K}_2 * \ln[(1 + e^{-\varepsilon^{2s}/T})(1 + e^{-\varepsilon^{-2s}/T})] + 2\bar{K}_1 * \ln(1 + e^{-\varepsilon^{ns}/T}) \} \end{aligned} \quad (4.5)$$

At  $T=0$ , we see from (4.5) that

$$\varepsilon_0^{\pm 2s} = \frac{3}{\sqrt{2} \cosh 3\lambda \mp 1}, \quad \varepsilon_0^{ns} = \frac{3}{\cosh 3\lambda} \quad (4.6)$$

and from (4.3) the free energy is  $f = \tilde{e}_0$ . These are the order-one excitations and ground-state energy for the antiferromagnetic Hamiltonian found in refs. 28 and 5.

Again we compute the low-temperature limit of the entropy, but now we rescale the integral equations by changing variables to  $\lambda \rightarrow \lambda - \frac{1}{3} \ln T$ . Defining  $\phi^\beta(\lambda) \equiv \varepsilon(\lambda - \frac{1}{3} \ln T)/T$ , we have

$$\begin{aligned} \phi^{2s} = \phi^{-2s} &\simeq 2 \times \frac{3}{\sqrt{2}} e^{-3\lambda} + T \{ K_1 * \ln[(1 + e^{-\phi^{2s}})(1 + e^{-\phi^{-2s}})] \\ &+ K_2 * \ln(1 + e^{-\phi^{ns}}) \} \tag{4.7} \\ \phi^{ns} &\simeq 2 \times 3e^{-3\lambda} + T \{ K_2 * \ln[(1 + e^{-\phi^{2s}})(1 + e^{-\phi^{-2s}})] \\ &+ 2K_1 * \ln(1 + e^{-\phi^{ns}}) \} \end{aligned}$$

so that  $2s$  and  $-2s$  are symmetric in this limit, which was a feature seen in ref. 3. Again, differentiating (4.7) with respect to  $\lambda$  and comparing to the density equations (4.1) rescaled as  $\lambda \rightarrow \lambda - \frac{1}{3} \ln T$ , we see that

$$\tilde{\rho}_p^\beta = -\frac{T}{3\pi} \frac{d\phi^\beta}{d\lambda} \frac{1}{1 + e^{\phi^\beta}}, \quad \beta = 2s, -2s, ns \tag{4.8}$$

The entropy is calculated as in (3.18). From (4.7) we find the limits  $\phi^\alpha(\pm \infty)$ ,

$$\begin{aligned} \phi^{2s}(-\infty) = \phi^{-2s}(-\infty) = \phi^{ns}(-\infty) &= \infty \\ \phi^{-2s}(\infty) = \phi^{2s}(\infty) = \ln 2, \quad \phi^{ns}(\infty) &= \ln 3 \end{aligned} \tag{4.9}$$

The entropy is

$$S \simeq -\frac{2T}{3\pi} \left[ 2 \int_{\infty}^{\ln 2} g(\phi) d\phi + \int_{\infty}^{\ln 3} g(\phi) d\phi \right] \tag{4.10}$$

which, using the dilogarithmic identities in the Appendix, gives

$$S \simeq \frac{2\pi T}{9} \tag{4.11}$$

This, with  $v = 3/2$ ,<sup>(5)</sup> gives a central charge  $c = 1$ , which is the central charge of the conformal limit of the model.<sup>(20,15)</sup> Again, this verifies that the limits (1.1) and (1.4) commute, and there are no additional length scales. We see that the fact that  $\rho_r^\alpha$  tends to decrease with increasing  $\rho_p^\alpha$  causes the central charge to be smaller than the value  $3/2$  one would expect for pure fermions.

## 5. $Q = \pm 1$

In the thermodynamic limit, quantities which are not of order  $M$  are irrelevant to the calculation. Therefore, we do not expect the value of  $Q$  to affect the thermodynamic equations. This is indeed the case.

In ref. 9 the counting rules for  $Q = \pm 1$  were found. It was shown that the counting rules depended on the value of the numbers  $m_{++}$  and  $m_{-+}$ , where

$$m_{++} - m_{-+} = 0, \pm 1 \quad (5.1)$$

Again we note that this difference is not of order  $M$  and we do not expect it to change the thermodynamic equations. The sum rules (2.16) for this sector are modified to

$$\begin{aligned} m_+ &= 2m_{ns} + 4m_{-2s} + m_{-+} + m_{++} \\ M - 1 &= m_{2s} + 2m_{ns} + 3m_{-2s} + m_{-+} + m_{++} \end{aligned} \quad (5.2)$$

This is only different from the  $Q = 0$  sector by a term of order one, due to Eq. (5.1) and the fact that  $m_{-+} = m_-$ . Therefore to order  $M$ , the sum rules are identical to (2.16).

For  $m_{++} - m_{-+} = \pm 1$ , the completeness rules for  $Q = 1$  are the same as for  $Q = 0$ . For  $m_{++} - m_{-+} = 0$ , there is spectrum doubling: The integers  $I_j^{2s}$  are shifted from those of  $+$  by  $\pm 1/2$ , and those of  $-2s$  are shifted from  $I_j^-$  in the same way, both signs giving the same energy level. The shift does not affect the thermodynamic limit of Eqs. (2.15). The spectrum doubling gives rise to an additive term of order one in the entropy (which counts the number of states). Since the entropy is of order  $M$ , again this term is not relevant in the thermodynamic limit. We conclude therefore that this sector is identical to  $Q = 0$ .

In ref. 3 the difference in counting rules for the sectors  $Q = \pm 1$  gave rise to different branching functions in the modular invariant partition function from those of  $Q = 0$ . However, each term in the partition function gives the same specific heat, due to the modular invariance property. The specific heat is found from the limit  $q \rightarrow 1$  of the partition function of ref. 3. However, the partition function is invariant under modular transformations, where, if  $q = \exp(2\pi i\tau)$ , the transformation  $\tau \rightarrow -1/\tau$  leaves the partition function invariant, so the specific heat is obtained from the  $q \rightarrow 0$  limit of the partition function. The same transformation sends each branching function into a linear combination of all other branching functions. Therefore, each branching function has the same  $q \rightarrow 1$  behavior. In the thermodynamic calculation we do not see this difference between the sectors  $Q = 0$  and  $Q = \pm 1$ .



## 6. DISCUSSION

In ref. 3 the order-one excitations (4.6) were used to compute the partition function of the antiferromagnetic chain in the limit (1.1). This is identical to the modular invariant partition function of the conformal limit of the model<sup>(17,35)</sup> and so gives the same specific heat as (4.11). For the computation in ref. 3 it was necessary to consider the way the maximum integer  $I_{\max}^{\alpha}$  of the finite-size system in (2.15) changed with the number of particles in the system. This is because those integers correspond to energies close to zero,  $e(P) \sim 0$ , or in the language used here, the large- $\lambda$  behavior of the energies, which is the region which contributes to the specific heat at low temperature. The maximum integers vary, at finite  $M$ , as more particles are added to the system. In ref. 3 this phenomenon was referred to as an *infrared momentum restriction*. This represents counting rules for the excitations beyond the fermionic exclusion rule, and is the phenomenon responsible for the central charge being different from that of fermions. In the case of the antiferromagnetic chain discussed in ref. 3, the infrared restriction was repulsive: fewer states were available as the number of excitations was increased than would be available for fermions. In the case of the ferromagnetic chain, both repulsive and attractive infrared restrictions are present, but the total infrared momentum restriction is attractive.

In the thermodynamic limit, we discard the information about the maximum integers  $I_j^{\alpha}$ . Nevertheless, the integral equations (4.5) contain the information about the way the density of available states depends on the particle density, represented by the density equations (4.1). This information enables correct counting of states, using the entropy. We see that this information gives the same specific heat as the counting of ref. 3, but, as we saw in Section 5, does not show the difference between the different  $Q$  sectors.

When computing the low-temperature specific heat, the region of  $\lambda$  which contributes to the free energy as  $T \rightarrow 0$  is the  $\lambda \sim \frac{1}{6} \ln T$  limit in the ferromagnetic case, and the  $\lambda \sim \frac{1}{3} \ln T$  limit in the antiferromagnetic case. This limit is the  $P \rightarrow 0$  limit,<sup>(5)</sup> which corresponds to the lowest-lying order-one excitations in ref. 3. The careful counting of states there is paralleled in the computation here by the rescaling of the integral equations and the density equations in the limit  $T \rightarrow 0$  and  $\lambda \rightarrow \infty$ .

Finally, we note that these computations are related to the thermodynamic Bethe ansatz method of ref. 36, a point which is discussed in some detail in ref. 22.

## APPENDIX. EXPRESSION OF ENTROPY VIA DILOGARITHMS

The Rogers dilogarithm is defined as<sup>(37)</sup>

$$L(x) = -\frac{1}{2} \int_0^x df \left( \frac{\ln(1-f)}{f} + \frac{\ln(f)}{1-f} \right) \quad (\text{A.1})$$

Making a change of variables in the expression for the entropy (3.18) to  $f = 1/1 + e^\phi$ , we can express the entropy (3.18) in terms of  $L(x)$ ,<sup>(15)</sup>

$$S \simeq -\frac{2T}{3\pi} \left( 2L\left(\frac{1}{3}\right) + L\left(\frac{1}{4}\right) - L(1) - 2L\left(\frac{3-\sqrt{5}}{2}\right) \right) \quad (\text{A.2})$$

We use the identity on Rogers dilogarithms<sup>(32)</sup>

$$\sum_{k=2}^{n-2} L\left(\frac{\sin^2(\pi/n)}{\sin^2(k\pi/n)}\right) = \frac{2(n-3)}{n} L(1) \quad (\text{A.3})$$

where  $L(1) = \pi^2/6$ .<sup>(38)</sup> Using this identity with  $n = 6$ , we find that

$$2L(1/3) + L(1/4) = L(1) \quad (\text{A.4})$$

and, with  $n = 5$ , we see that<sup>(38)</sup>

$$2L\left(\frac{3-\sqrt{5}}{2}\right) = 2\frac{\pi^2}{15} \quad (\text{A.5})$$

Therefore Eq. (A.2) gives

$$S \simeq \frac{4T}{3\pi} L\left(\frac{3-\sqrt{5}}{2}\right) = \frac{4\pi T}{45} \quad (\text{A.6})$$

In the antiferromagnetic case, we again use (A.4) with  $n = 6$ , and entropy is

$$S \simeq \frac{4T}{3\pi} \left[ 2L\left(\frac{1}{3}\right) + L\left(\frac{1}{4}\right) \right] = \frac{4T}{3\pi} L(1) = \frac{2\pi T}{9} \quad (\text{A.7})$$

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